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A NEW SOLUTION TO THE PROBLEM OF THE THEORY OF ELASTICITY IN STRESSES ABOUT A ROTATING ROTATION ELLIPSOID

КЕРНЕУДЕГІ АЙНАЛМАЛЫ ЭЛЛИПСОИД АЙНАЛУЫ ТУРАЛЫ СЕРПІМДІЛІК ТЕОРИЯСЫНЫҢ ЕСЕБІНІҢ ЖАҢА ШЕШІМІ

НОВОЕ РЕШЕНИЕ ЗАДАЧИ ТЕОРИИ УПРУГОСТИ В НАПРЯЖЕНИЯХ О ВРАЩАЮЩЕМСЯ ЭЛЛИПСОИДЕ ВРАЩЕНИЯ

Abstract. *An exact semi-inverse solution of the fundamental axisymmetric spatial problem of the mechanics of an elastically deformable solid medium on the uniform rotation around a fixed axis of a weightless ellipsoid made of a homogeneous isotropic material obeying the generalized Hooke law, taking into account the mass centrifugal forces of inertia was obtained based on the two-dimensional equilibrium equations and modified conditions of deformations continuity. The uncertainty (incorrectness) associated with the multivariance of the approximate solution of the problem by the inverse method, depending on the choice of approximating expressions for the stress function of the homogeneous biharmonic Love's equation and the selection of the corresponding internal force factors for the particular solution has been eliminated. Compact and physically justified stress formulas in the form of four quadratic polynomials satisfying the system of resolving differential equations and all boundary conditions on the ellipsoid surface have been derived. A numerical evaluation of the known theory has been carried out, in comparison with the proposed new solution for three values of the Poisson's ratio, covering the permissible range from 0 to 0.5, for the special case of a rotating ball (when the semiaxis of the ellipse are equal). In the process of this comparison, a sufficiently large error has been established, even from an engineering point of view, of the existing mathematical model, which, according to the extreme main stresses acting in the center of a spherical body, ranges from 16.22% to 22.82%. The great importance and necessity of the practical application of the developed classical mechanical and mathematical model to the high-precision calculation of the strength of the rapidly rotating turbine disks of elliptical profile in order to increase their guaranteed load-bearing capacity and reliability during the operation has been noted and illustrated.*
Keywords*: stress; strain; angular velocity; elasticity; ellipsoid; axial symmetry.*

Аңдатпа. *Екі өлшемді тепе-теңдік теңдеулеріне және деформациялардың өзгермейтіндігінің модификацияланған жағдайларына сүйене отырып, екінші дәрежелі қарапайым көпмүшелерді қолдана отырып, массалық центрден тепкіш инерция күштерін ескере отырып, Гуктың жалпыланған заңына бағынатын біртекті изотропты материалдан жасалған салмақсыз эллипсоидтың қозғалмайтын өсі бойынша біркелкі айналу туралы серпімді-деформацияланатын қатты орта механикасының іргелі өсті симметриялық кеңістіктік есебінің нақты жартылай қайта шешімі алынды. Эллиптикалық дененің айналу динамикасының неғұрлым күрделі* *математикалық сипаттамасынан айырмашылығы, біртекті бигармоникалық Ляв теңдеуінің кернеу функциясы үшін жуықтайтын өрнектерді таңдауға және жеке шешім үшін тиісті ішкі күш факторларын таңдауға байланысты есепті кері әдіспен жуық шешудің көп нұсқалылығымен байланысты белгісіздік (қате) жойылды. Дифференциалдық теңдеулер жүйесін және эллипсоид бетіндегі барлық шекаралық жағдайларды қанағаттандыратын төрт квадраттық көпмүшелік түрінде ықшам және физикалық негізделген кернеу формулалары алынды. 0-ден 0,5-ке дейінгі рұқсат етілген диапазонды қамтитын Пуассон коэффициентінің үш мәні үшін айналмалы доптың нақты жағдайы үшін (эллипстің жартылай өсіне тең болған кезде) ұсынылған жаңа шешіммен салыстырғанда белгілі теорияны сандық бағалау жүргізілді. Бұл салыстыру барысында, тіпті инженерлік-техникалық тұрғыдан алғанда, қолданыстағы математикалық модель, сфералық дененің ортасында әрекет ететін экстремалды негізгі кернеулер бойынша 16,22 %-дан 22,82 %-ға дейін ауытқитын айтарлықтай үлкен қателік анықталды. Пайдалану кезінде олардың кепілдік берілген көтеру қабілеті мен сенімділігін арттыру мақсатында эллиптикалық бейіндегі жылдам айналатын турбиналық дискілердің беріктігіне жоғары дәлдікті есептеуге әзірленген классикалық механикалық-математикалық модельді практикалық қолданудың үлкен маңыздылығы мен қажеттілігі көрнекі мысалмен атап өтілді және көрсетілді.*

***Түйін сөздер**: керне; деформация; бұрыштық жылдамдық; серпімділік; эллипсоид; аксиальді симметрия.*

***Аннотация.** Базируясь на двумерных уравнениях равновесия и модифицированных условиях неразрывности деформаций получено, с применением простых многочленов второй степени, точное полуобратное решение фундаментальной осесимметричной пространственной задачи механики упруго-деформируемой твердой среды о равномерном вращении вокруг неподвижной оси невесомого эллипсоида из однородного изотропного материала, подчиняющегося обобщенному закону Гука, с учетом массовых центробежных сил инерции. В отличие от существующего более сложного математического описания динамики вращения эллиптического тела, устранена неопределенность (некорректность), связанная с многовариантностью приближенного решения задачи обратным методом в зависимости от выбора аппроксимирующих выражений для функции напряжений однородного бигармонического уравнения Лява и подбора соответствующих внутренних силовых факторов для частного решения. Выведены компактные и физически обоснованные формулы напряжений в виде четырех квадратичных полиномов, удовлетворяющих системе разрешающих дифференциальных уравнений и всем граничным условиям на поверхности эллипсоида. При трех значениях коэффициента Пуассона, охватывающих допустимый диапазон от 0 до 0,5, для частного случая вращающегося шара (когда равны полуоси эллипса) проведена численная оценка известной теории, по сравнению с предложенным новым решением. В процессе этого сопоставления установлена достаточно большая погрешность, даже с инженерно-технической точки зрения, существующей математической модели, которая по экстремальным главным напряжениям, действующим в центре сферического тела, колеблется от 16,22 % до 22,82 %. Отмечена и проиллюстрирована наглядным примером большая значимость и необходимость практического применения разработанной классической механико-математической модели к высокоточному расчету на прочность быстро вращающихся турбинных дисков эллиптического профиля с целью повышения их гарантированной несущей способности и надежности при эксплуатации.*

***Ключевые слова:** напряжение; деформация; угловая скорость; упругость; эллипсоид; аксиальная симметрия.*

Introduction. The being considered classical spatial problem of the linear mechanics of an elastically deformable homogeneous isotropic body in the form of a rotating one with a constant angular velocity $\omega = const$ of a weightless ellipsoid of rotation has a fundamental solution [1-3], the basis of which is [1-14] (Fig. 1):

− equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{\sigma_r - \sigma_\theta}{r} + \dfrac{\partial \tau}{\partial z} + \dfrac{\gamma}{g} \cdot \omega^2 \cdot r = 0, & (1) \\ \dfrac{\partial \tau}{\partial r} + \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\tau}{r} = 0; & (2) \end{cases}$$

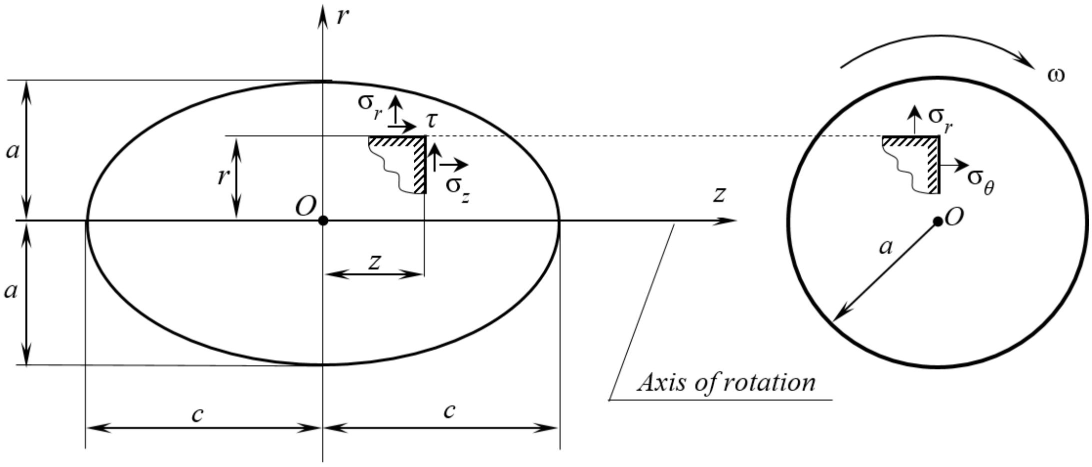

Figure 1. Calculation layout of an ellipsoid rotating with $\omega = const$ angular velocity

−four conditions for the deformations compatibility [1-5, 8]

$$\left.\begin{aligned}
&\nabla^2\sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\mu}\cdot\frac{\partial^2\Sigma}{\partial r^2} = -\frac{2\gamma\cdot\omega^2}{g(1-\mu)},\\
&\nabla^2\sigma_\theta + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\mu}\cdot\frac{1}{r}\cdot\frac{\partial\Sigma}{\partial r} = -\frac{2\gamma\cdot\omega^2}{g(1-\mu)},\\
&\nabla^2\sigma_z + \frac{1}{1+\mu}\cdot\frac{\partial^2\Sigma}{\partial r^2} = -\frac{2\mu\cdot\gamma\cdot\omega^2}{g(1-\mu)},\\
&\nabla^2\tau - \frac{\tau}{r^2} + \frac{1}{1+\mu}\cdot\frac{\partial^2\Sigma}{\partial r\cdot\partial z} = 0,
\end{aligned}\right\} \tag{3}$$

−here $\Sigma = \sigma_r + \sigma_\theta + \sigma_z$ – the sum of normal stresss;

−love 's biharmonic equation [1-5]

$$\nabla^4\varphi = \nabla^2\nabla^2\varphi = 0, \tag{4}$$

equivalent to a homogeneous system (1) – (3), if, to get a general solution, using differential dependencies

$$\left.\begin{aligned}
\sigma_{r0} &= \frac{\partial}{\partial z}\left(\mu\cdot\nabla^2\varphi - \frac{\partial^2\varphi}{\partial r^2}\right),\\
\sigma_{\theta 0} &= \frac{\partial}{\partial z}\left(\mu\cdot\nabla^2\varphi - \frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}\right),\\
\sigma_{z0} &= \frac{\partial}{\partial z}\left((2-\mu)\cdot\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right),\\
\tau_0 &= \frac{\partial}{\partial r}\left((1-\mu)\cdot\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right);
\end{aligned}\right\} \tag{5}$$

− partial integrals, when, representing functions that are even in argument *r* [1,2]

$$\left.\begin{aligned}
\sigma_{r\text{ч}} &= \sigma_{r\text{ч}}(r) = -\frac{\gamma\cdot\omega^2}{8g}\cdot\frac{3-2\mu}{1-\mu}\cdot r^2,\\
\sigma_{\theta\text{ч}} &= \sigma_{\theta\text{ч}}(r) = -\frac{\gamma\cdot\omega^2}{8g}\cdot\frac{1+2\mu}{1-\mu}\cdot r^2,\\
\sigma_{z\text{ч}} &= \sigma_{z\text{ч}}(r) = -\frac{\gamma\cdot\omega^2}{2g}\cdot\frac{\mu}{1-\mu}\cdot r^2, \quad \tau_{\text{ч}} = 0;
\end{aligned}\right\} \tag{6}$$

− two homogeneous boundary condition [1, 2, 6]

$$\beta\cdot \mathrm{r}\cdot\sigma_{\mathrm{r}} + \mathrm{z}\cdot\tau = 0, \tag{7}$$

$$\beta\cdot \mathrm{r}\cdot\tau + \mathrm{z}\cdot\sigma_{\mathrm{z}} = 0, \tag{8}$$

since the surface of the ellipsoid is free from external load (Fig. 1), where, according to the equation of an ellipse with c, a semi-axes [15]

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{9}$$

it is necessary to replace the z variable by taking

$$z^2 = c^2 - \beta\cdot r^2. \tag{10}$$

In the ratios (1) – (8): $\sigma_r, \sigma_\theta, \sigma_z, \tau$ are accordingly, the required normal (σ_r – radial, σ_θ – tangential, σ_z – (azimuthal) axial) and τ tangent of tension (Fig. 1);

∇^2 is Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{11}$$

in *r, z* cylindrical coordinates [1-5, 9, 10, 16] for conditions of the axial symmetry of the rotation body [1,2] when volumetric and surface forces (in this case, the inertial radial mass load $\frac{\gamma}{g}\cdot\omega^2\cdot r$ – (1)) do not depend on the azimuth angle θ; ω is an angular velocity (the frequency of the ellipsoid rotation);

γ, μ are correspondingly, the specific gravity (γ) and the Poisson 's ratio (μ) of the rotation body material; $g = 9810\left(\frac{mm}{s^2}\right)$ is the acceleration of gravity; β – the square of the ratio of the semiaxes of the ellipsoid (Fig. 1)

$$\beta = \frac{c^2}{a^2}, \tag{12}$$

which can be flattened when β<1 (*c*<*a*) as well as elongated along the axis *z*, when β>1 (*c*>*a*), and in a special case β=1 there will be a sphere of radius

$$\mathrm{c} = \mathrm{a} = \mathrm{R}. \tag{13}$$

Literature Review. The existing rather complex approximate solution of this problem by the inverse method [4, 8, 14, 16] it is based on the use of formulas (4) – (11) and sums of Legendre polynomials [1-4], including: four constants, two polynomials of the 3rd degree and two of the 5th degree describing the resolving function $\varphi = \varphi(r, z)$ of differential equation (4). As it is known [5, 16], the accuracy of this method depends, first of all, on the successful (qualitative) selection of approximating analytical expressions adequately characterizing the stress tensor [1-3, 10]. And here a variant approach is possible in defining a particular solution (6) [1, 2, 4], as well as when specifying Legendre power polynomials of the first degree [1-3, 17] which will naturally lead to different end results, that is, to non-compliance with the uniqueness theorem of the solution of the elasticity theory problem [1-5, 10, 18, 19].

Materials and methods of research. To eliminate these shortcomings including the incorrectness associated with multivariance, the authors of this paper propose a fundamentally new theoretical algorithm for more accurate mathematical modeling of the dynamic process under consideration based on its practical implementation on the semi-inverse Saint-Venant method [4, 5, 13, 14, 16], which is currently one of the most effective conceptual techniques for the analytical solution of complex fundamental and applied problems of deformable solid mechanics [14, 18, 19, 20, 21].

Having the purpose of subsequent comparative evaluation of the new solution the authors present an example of a special case of specific known approximate formulas and numerical values of the greatest normal stresses [1, 2] arising in the center of a rotating sphere $\mathrm{r} = \mathrm{z} = 0$, when $\mathrm{c} = \mathrm{a} = \mathrm{R}$, a $\beta = 1$ ((12), (13) and Fig. 1), where $\tau = 0$, with coefficients $\mu = 0;\ 0{,}3;\ 0{,}5$, covering the maximum permissible range of their changes $0 \le \mu \le 0{,}5$ [22, 23-26]:
μ=0 is stopper [23]

$$\left.\begin{aligned}
&\sigma(0{,}0) = \sigma_r(0{,}0) = \sigma_\theta(0{,}0) = \frac{\gamma \cdot \omega^2 \cdot \mathrm{R}^2}{10 \cdot \mathrm{g} \cdot (1-\mu) \cdot (7+5\mu)} \cdot \{(3-2\mu) \cdot (6+5\mu) + \\
&+ \mu \cdot (11+10\mu) + 2 \cdot [3 - 2\mu - \mu \cdot (5\mu + 4)]\} = \\
&= \frac{\gamma \cdot \omega^2 \cdot \mathrm{R}^2}{10 \cdot \mathrm{g} \cdot (1-0) \cdot (7+5 \cdot 0)} \cdot \{(3-2\cdot 0) \cdot (6+5 \cdot 0) + 0 \cdot (11+10 \cdot 0) + \\
&+ 2 \cdot [3 - 2 \cdot 0 - 0 \cdot (5 \cdot 0 + 4)]\} = 0{,}343 \cdot \frac{\gamma \cdot \omega^2 \cdot \mathrm{R}^2}{\mathrm{g}},
\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}\sigma_z(0{,}0) &= \frac{\gamma\cdot\omega^2\cdot R^2}{10g(1-\mu)\cdot(7+5\mu)}\cdot[4\mu-6+2\mu(5\mu+4)] = \\ &= \frac{\gamma\cdot\omega^2\cdot R^2}{10g(1-0)\cdot(7+5\cdot 0)}\cdot[4\cdot 0-6+2\cdot 0\cdot(5\cdot 0+4)] = -0{,}0857\cdot\frac{\gamma\cdot\omega^2\cdot R^2}{g};\end{aligned}\right\} \tag{15}$$

μ=0,3 are deformable aluminum alloys, titanium and its alloys [26], steel, glass [22]

$$\sigma(0{,}0) = \sigma_r(0{,}0) = \sigma_\theta(0{,}0) = 0{,}418\cdot\frac{\gamma\cdot\omega^2\cdot R^2}{g}, \tag{16}$$

$$\sigma_z(0{,}0) = -0{,}0264\cdot\frac{\gamma\cdot\omega^2\cdot R^2}{g}; \tag{17}$$

μ=0,5 – is rubber [24, 25]

$$\sigma(0{,}0) = \sigma_r(0{,}0) = \sigma_\theta(0{,}0) = 0{,}474\cdot\frac{\gamma\cdot\omega^2\cdot R^2}{g}, \tag{18}$$

$$\sigma_z(0{,}0) = 0{,}0526\cdot\frac{\gamma\cdot\omega^2\cdot R^2}{g}. \tag{19}$$

Using the algorithm of the semi-inverse method the tangential force component is found from equation (1)

$$\bar{\sigma}_\theta = \frac{\gamma\cdot\omega^2\cdot r^2}{g} + r\cdot\frac{\partial\bar{\sigma}_r}{\partial r} + \sigma_r + r\cdot\frac{\partial\bar{\tau}}{\partial z} \tag{20}$$

and substituted into the first modified condition for the deformations compatibility expressed in the terms of stresses $\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_z$ [6, 12, 13]

$$r\cdot\frac{\partial}{\partial r}[\bar{\sigma}_\theta - \mu(\bar{\sigma}_z + \bar{\sigma}_r)] + (1+\mu)\cdot(\bar{\sigma}_\theta - \bar{\sigma}_r) = 0, \tag{21}$$

by converting it to the shape

$$\frac{\partial}{\partial r}\cdot\left[r\cdot\frac{\partial\bar{\sigma}_r}{\partial r} + r\cdot\frac{\partial\bar{\tau}}{\partial z} - \mu(\bar{\sigma}_z + \bar{\sigma}_r)\right] + (1+\mu)\cdot\left(\frac{\partial\bar{\sigma}_r}{\partial r} + \frac{\partial\bar{\tau}}{\partial z}\right) = -(3+\mu)\cdot\frac{\gamma\cdot\omega^2}{g}\cdot r, \tag{22}$$

including the tangent component $\bar{\tau}$.

Taking into account the axial symmetry of the calculation model of Figure 1 and in accordance with [1, 2] it is supposed that:

$$\bar{\sigma}_r = \bar{\sigma}_r(r,z) = C_1 + C_2\cdot r^2 + C_3\cdot z^2, \tag{23}$$

$$\bar{\tau} = \bar{\tau}(r,z) = C_4\cdot r\cdot z; \tag{24}$$

when $C_1, \ldots, C_4$ are constants to be determined.

The horizontal line above the alphabetic characters $\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_z, \bar{\tau}$ is a distinctive feature of the listed parameters belonging to the improved computational and theoretical model.

The tangential $\bar{\tau}$ is excluded from equation (2) using substitution (24) and after its solution [17] $\bar{\sigma}_z$ is calculated with the accuracy up to arbitrary integration $f(r)$:

$$\left.\begin{aligned}&\frac{\partial\bar{\sigma}_z}{\partial z} = -\frac{\partial\bar{\tau}}{\partial r} - \frac{\bar{\tau}}{r} = -2C_4\cdot z, \quad\Rightarrow\\ &\bar{\sigma}_z = -C_4\cdot z^2 + f(r).\end{aligned}\right\} \tag{25}$$

Taking into account the expressions (24), (25) the boundary equality is revealed (8) when

$z^2 = c^2 - \beta \cdot r^2$, according to (10):

$$C_4 \cdot r^2 - C_4 \cdot z^2 + f(r) = C_4 \cdot r^2 - C_4 \cdot c^2 + C_4 \cdot \beta \cdot r^2 + f(r) = 0, \tag{26}$$

from where the following is got

$$f(r) = C_4 \cdot c^2 - C_4(1+\beta) \cdot r^2. \tag{27}$$

Taking into account (25) and (27) there will be

$$\bar{\sigma}_z = \bar{\sigma}_z(r,z) = C_4 \cdot \left[c^2 - z^2 - (1+\beta) \cdot r^2\right]. \tag{28}$$

Next, we specify the tangential (azimuthal) component $\bar{\sigma}_\theta$ internal force factors, using the ratios (20), (23), (24), and in the end it is found

$$\bar{\sigma}_\theta = \bar{\sigma}_\theta(r,z) = \frac{\gamma \cdot \omega^2}{g} \cdot r^2 + C_1 + 3 \cdot C_2 \cdot r^2 + C_3 \cdot z^2 + C_4 \cdot r^2. \tag{29}$$

The first boundary condition (7) the given approximating polynomials (23), (24) and formula (10), results in a functional sum

$$\beta \cdot C_1 + \beta \cdot C_2 \cdot r^2 + C_3 \cdot \beta \cdot c^2 - C_3 \cdot \beta^2 \cdot r^2 + C_4 \cdot c^2 - C_4 \cdot \beta \cdot r^2 = 0, \tag{30}$$

equal to zero, which decays into two homogeneous algebraic equations

$$\beta \cdot C_1 + C_3 \cdot \beta \cdot c^2 + C_4 \cdot c^2 = 0, \tag{31}$$

$$C_2 - \beta \cdot C_3 - C_4 = 0 \tag{32}$$

and an additional dependency

$$C_1 = -C_2 \cdot \frac{c^2}{\beta} = -C_2 \cdot a^2, \tag{33}$$

after multiplying (32) by c^2 and slow addition (31), (32).

Continuing to solve the problem, we exclude from the formulas (23), (24), (28), (29) constants C_1, C_4, using the ratios (32), (33). The result of this procedure will be the following analytical expressions:

$$\bar{\sigma}_r = \bar{\sigma}_r(r,z) = -C_2\left(a^2 - r^2\right) + C_3 \cdot z^2, \tag{34}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\theta(r,z) = -C_2\left(a^2 - 4r^2\right) + C_3\left(z^2 - r^2\right) + \frac{\gamma}{g} \cdot \omega^2 \cdot r^2, \tag{35}$$

$$\bar{\sigma}_z = \bar{\sigma}_z(r,z) = (C_2 - \beta \cdot C_3) \cdot \left[c^2 - z^2 - (1+\beta) \cdot r^2\right], \tag{36}$$

$$\bar{\tau} = \bar{\tau}(r,z) = (C_2 - \beta \cdot C_3) \cdot r \cdot z \tag{37}$$

In the center of an elliptical body where $r = z = 0$ due to the axial symmetry of the drawing model 1 [1-3],

$$\left.\begin{aligned} &\bar{\sigma}(0,0) = \bar{\sigma}_r(0,0) = \bar{\sigma}_\theta(0,0) = -C_2 \cdot a^2 \\ &\bar{\tau}(0,0) = 0, \end{aligned}\right\} \tag{38}$$

and an axial stress is

$$\bar{\sigma}_z(0,0) = (C_2 - \beta \cdot C_3) \cdot c^2. \tag{39}$$

Substituting (34) – (37) into the transformed continuity condition (22) and a similar second equation of the compatibility of deformations [4, 6, 12, 13]

$$\frac{\partial}{\partial \mathrm{r}}\left[\bar{\sigma}_\mathrm{z} - \mu(\bar{\sigma}_\mathrm{r} + \bar{\sigma}_\theta)\right] + \frac{\partial^2}{\partial \mathrm{z}^2}\left[\bar{\sigma}_\theta - \mu(\bar{\sigma}_\mathrm{z} + \bar{\sigma}_\mathrm{r})\right] - 2(1+\mu)\frac{\partial \bar{\tau}}{\partial \mathrm{z}} = 0, \tag{40}$$

the canonical algebraic system is got [15, 17]

$$\left.\begin{aligned} a_{11}\cdot C_2 + a_{12}\cdot C_3 = b_1, \\ a_{21}\cdot C_2 + a_{22}\cdot C_3 = b_2, \end{aligned}\right\} \tag{41}$$

in which $a_{11},\ a_{12},\ a_{21},\ a_{22}$ are the coefficients for unknown $C_2,\ C_3$

$$a_{11} = 2+\beta+5\cdot\mu\ ,\qquad a_{12} = -[\beta\cdot(2+\beta)+1], \tag{42}$$

$$a_{21} = 9+7\cdot\mu\ ,\qquad a_{22} = -\beta\cdot(3+5\cdot\mu), \tag{43}$$

and $b_1,\ b_2$ are free members

$$b_1 = -\mu\cdot\frac{\gamma}{g}\cdot\omega^2\ ,\qquad b_2 = -(3+\mu)\cdot\frac{\gamma}{g}\cdot\omega^2\,. \tag{44}$$

Compact form of being solved equations (41) is like this [15]:

$$C_2 = \frac{D_2}{D}\ ,\qquad C_3 = \frac{D_3}{D}, \tag{45}$$

when $D,\ D_2,\ D_3$ are determinants of the second order [15, 17]

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12} = \\ = -\beta\cdot(2+\beta+5\mu)\cdot(3+5\mu)+(9+7\mu)\cdot[\beta\cdot(2+\beta)+1], \tag{46}$$

$$D_2 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1\cdot a_{22} - b_2\cdot a_{12} = \\ = \frac{\gamma}{g}\cdot\omega^2\cdot\{\mu\cdot\beta\cdot(3+5\mu)-(3+\mu)\cdot[\beta\cdot(2+\beta)+1]\}, \tag{47}$$

$$D_3 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11}\cdot b_2 - a_{21}\cdot b_1 = \\ = \frac{\gamma}{g}\cdot\omega^2\cdot[-(2+\beta+5\mu)\cdot(3+\mu)+(9+7\mu)\cdot\mu]. \tag{48}$$

Table 1 shows the results of constants $C_2,\ C_3$ calculation according to (45) – (48), and extreme main stresses $\bar{\sigma}_r(0{,}0)=\bar{\sigma}_\theta(0{,}0)=\bar{\sigma}(0{,}0),\ \bar{\sigma}_z(0{,}0)$ based on (38), (39) for the sphere with the radius $R=a=c,$ when $\beta=\dfrac{c^2}{a^2}=1$ for $\mu=0;\ 0{,}3;\ 0{,}5.$

Table 1. Calculated data by $C_2,\ C_3,\ \bar{\sigma}(0{,}0),\ \bar{\sigma}_z(0{,}0)$ definition

Required parameters	The value of the Poisson's ratio μ		
	0	0,3	0,5
D	27	23.75	19.75
$D_2\times\dfrac{g}{\gamma\cdot\omega^2}$	– 12	– 11.85	– 11.25

$D_3 \times \frac{g}{\gamma \cdot \omega^2}$	– 9	– 11,52	– 13
$C_2 \times \frac{g}{\gamma \cdot \omega^2}$	– 0,4444	– 0,4989	– 0,5696
$C_3 \times \frac{g}{\gamma \cdot \omega^2}$	– 0,3333	– 0,485	– 0,6582
$\overline{\sigma}(0,0) \times \frac{g}{\gamma \cdot \omega^2 \cdot R^2}$	– 0,4444	– 0,4989	– 0,5696
$\overline{\sigma}_z(0,0) \times \frac{g}{\gamma \cdot \omega^2 \cdot R^2}$	– 0,1111	– 0,0139	– 0,0886

Considering the general ratios (34) – (37), there are found the functions of the main stresses acting in sections of spherical body 1-1, 2-2, which $a = c = R$ $\left(\Rightarrow \beta = c^2 \cdot a^{-2} = 1\right)$, coinciding with its axes of symmetry, where $\overline{\tau}(0,z) = \overline{\tau}(r,0) = 0$ (Fig. 1 and 2):

−when $r = 0, \ -R \le z \le R$ (section 1-1)

$$\begin{cases} \overline{\sigma}_r(0,z) = \overline{\sigma}_\theta(0,z) = -C_2 \cdot R^2 + C_3 \cdot z^2, & (49) \\ \overline{\sigma}_z(0,z) = (C_2 - C_3) \cdot \left(R^2 - z^2\right); & (50) \end{cases}$$

−when $-R \le r \le R, \ z = 0$ (section 2-2)

$$\overline{\sigma}_r(r,0) = -C_2\left(R^2 - r^2\right), \qquad (51)$$

$$\overline{\sigma}_\theta(r,0) = -C_2\left(R^2 - 4r^2\right) - C_3 \cdot r^2 + \frac{\gamma \cdot \omega^2}{g} \cdot r^2, \qquad (52)$$

$$\overline{\sigma}_z(r,0) = (C_2 - C_3) \cdot \left(R^{2\cdot} - 4r^2\right) \qquad (53)$$

Using equations (49) – (53), $\mu = 0,3$ is got (Table 1):

−for section 1-1 $\left(r = 0, \ -R \le z \le R\right)$

$$\begin{cases} \overline{\sigma}_r(0,z) = \overline{\sigma}_\theta(0,z) = \frac{\gamma \cdot \omega^2}{g} \cdot \left(0,4989R^2 - 0,485z^2\right), & (54) \\ \overline{\sigma}_z(0,z) = -0,0139 \cdot \frac{\gamma \cdot \omega^2}{g} \cdot \left(R^2 - z^2\right), & (55) \end{cases}$$

−for section 2-2 $\left(-R \le r \le R, \ z = 0\right)$

$$\begin{cases} \overline{\sigma}_r(r,0) = 0,4989 \cdot \frac{\gamma \cdot \omega^2}{g} \cdot \left(R^2 - r^2\right), & (56) \\ \overline{\sigma}_\theta(r,0) = \frac{\gamma \cdot \omega^2}{g} \cdot \left[0,4989 \cdot \left(R^2 - 4r^2\right) + 1,485 \cdot r^2\right], & (57) \\ \overline{\sigma}_z(r,0) = -0,0139 \cdot \left(R^2 - 2r^2\right). & (58) \end{cases}$$

Having functional dependencies (54) – (58), their values are at characteristic points 0, 1, 2 of the sphere are determined. Table 2 shows the calculation results and Figure 2 illustrates plots of internal force parameters representing symmetric parabolas of the second degree.

Completing the computational process by illustrating this fundamental problem the Table 3 is made with final numerical data on the main greatest stresses $\sigma_r(0{,}0)=\sigma_\theta(0{,}0)=\sigma(0{,}0)$, $\sigma_z(0{,}0)$ for the sphere center (Fig. 2), found by the approximate inverse method [1,2] and presented by the results of calculations (14) – (19) comparing with a new exact solution $\bar{\sigma}_r(0{,}0)=\bar{\sigma}_\theta(0{,}0)=\bar{\sigma}(0{,}0)$, $\bar{\sigma}_z(0{,}0)$ using formulas (38), (39), (45) – (48) (Table 1) for three values of the Poisson 's ratio μ on a closed interval of its change $0 \le \mu \le 0{,}5$ [22, 23-26].

Considering the quantitative information in Table 3, it can be stated that at least in the special case of a spherical shape of a rotating body (Fig. 2) the well-known solution [1,2] is a rather rough approximation even from an engineering and technical point of view since it gives an error on the main, determining the bearing capacity of a solid, normal stress $\bar{\sigma}_r(0{,}0)=\bar{\sigma}_\theta(0{,}0)=\bar{\sigma}(0{,}0)\ge\bar{\sigma}_z(0{,}0)$ from 16,22 % for μ=0,3 to 22,82 %, when μ=0 [23]. At the same time, the results of the calculation of the secondary (not the main) axial component of internal forces $\sigma_z(0{,}0)$ have the largest margin of error, reaching 89,93% for some widely used metal materials and glass that have μ=0,3 [8, 13, 24, 26].

Table 2. Dimensionless calculated information on stresses (54) – (58) for points 0, 1, 2 of a spherical body at $\mu = 0{,}3$

Point No. (and its coordinates r, z)	Main stresss		
	$\bar{\sigma}_r \times \frac{g}{\gamma\cdot\omega^2\cdot R^2}$	$\bar{\sigma}_\theta \times \frac{g}{\gamma\cdot\omega^2\cdot R^2}$	$\bar{\sigma}_z \times \frac{g}{\gamma\cdot\omega^2\cdot R^2}$
0 (*r*=*z*=0)	0,4989	0,4989	-0,0139
1 (*r*=0, *z*=±*R*)	0,0139	0,0139	0
2 (*r*=±*R*, *z*=0)	0	-0,0117	0,0139

Table 3. The final calculated results of dimensionless stresses in the center of the spherical model

μ	0	0,3	0,5
$\bar{\sigma}(0{,}0)\times\frac{g}{\gamma\cdot\omega^2\cdot R^2}$	0,4444	0,4989	0,5696
$\sigma(0{,}0)\times\frac{g}{\gamma\cdot\omega^2\cdot R^2}$	0,343	0,418	0,474
Error of the solution [1,2], %	22,82	16,22	16,78
$\bar{\sigma}_z(0{,}0)\times\frac{g}{\gamma\cdot\omega^2\cdot R^2}$	-0,1111	-0,0139	0,0886
$\sigma_z(0{,}0)\times\frac{g}{\gamma\cdot\omega^2\cdot R^2}$	-0,0857	-0,0264	0,0526

Error of the solution [1,2]	22,86	89,93	40,63

(Fig. 2), according to the expressions (38), (39), (45) – (48), and their comparison with the approximate solution (14) – (19) [1, 2]

Results and discussion. In conclusion, the authors show a typical example of practical use of a new generalized fundamental solution got by them (34) – (37), (45) – (48) to evaluate the strengths of disk-shaped elliptical profile parts (Fig. 3), rotating at a circumferential speed of 400 m/s or more [25-27], in order to clarify the boundaries of the required load-bearing capacity. This task is one of the most important problems in the design, production and operation of gas turbine aircraft engines, steam and gas turbines, ultracentrifuges, etc. [4, 27-37], the most responsible structural elements of which are disks, and their destruction carries, as a rule, catastrophic consequences.

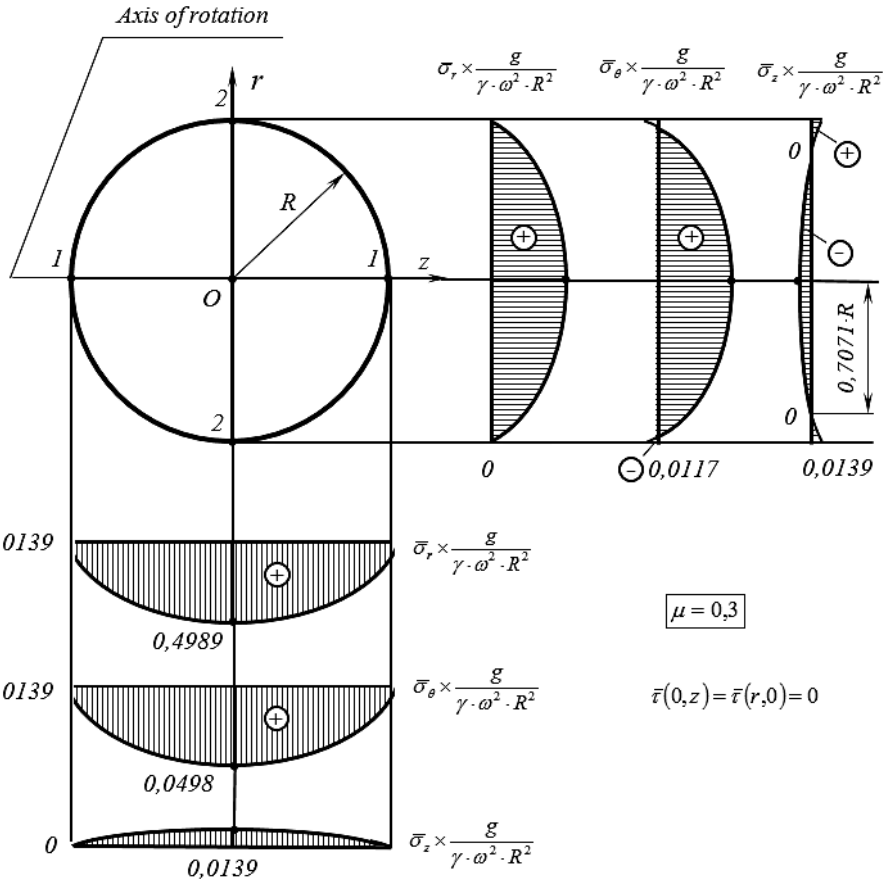

Figure 2. Graphical representation of extreme normal stresses for the axial 1-1 and equatorial 2-2 cross-sections of the ball, when $\mu = 0{,}3$

The currently existing approximate method of calculating disks is based on the theoretical dependencies of the plane stress state [4, 7, 10, 16, 18, 30, 35, 40] the possibility of using which is confirmed by the research of A. Stodola [31, 32] and Cree [36], when the bearing part of the structure, having a disc-shaped shape, is presented in the form of a thin round plate of variable thickness *h=h(r)* on the condition [31, 32] (Fig. 3)

$$\frac{h_0}{d} = \sqrt{\beta} = \frac{c}{a} \le \frac{1}{8}, \tag{59}$$

here $h_0 = h(0) = 2c = \max$ – the greatest value of the function $h(r)$; $d = 2a$ – the disc diameter.

Only if the boundary inequality (59) is observed, the engineering assessment of the reliability of the disk can be based on five known prerequisites (Fig. 3) [4, 27-32]: 1) radial σ_r and tangential σ_θ stresses are constant in thickness $h(r)$, that is, they do not depend on the variable *z*; 2) the components of internal forces axial $\sigma_z = 0$ and tangent $\tau = 0$ equal to zero; 3) the disk has a plane of symmetry coinciding with the coordinate axis *r*; 4) the disk is stretched by centrifugal forces of inertia [27]

$$N_i = N_i(r) = \frac{\gamma}{g} \cdot \omega^2 \cdot r^2 \cdot h, \tag{60}$$

arising from its rotation at a speed of $\omega = const$ which are assumed to be symmetric to the axis *z*, assuming that its own weight is negligible; 5) the material is homogeneous, solid, isotropic and obeys Hooke's law.

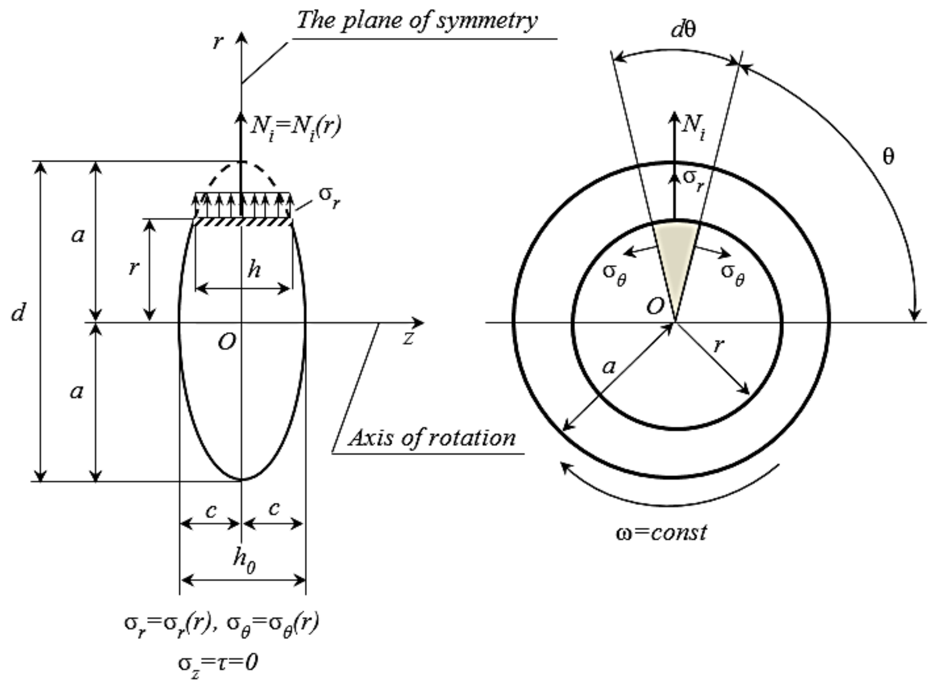

Figure 3. Simplified model for calculating stresses in turbine elliptical disks at $\sqrt{\beta} \le 0{,}125$ (see (12) and (49)) [27-32]

The application of the physical and mathematical theory presented in this article to the high-precision calculation of rotating elliptical disks becomes obvious and extremely relevant as a consequence of the above.

The analysis of the performed complex studies allows drawing the following *conclusions*:

1. An original solution in the components of the stress tensor of the fundamental multifunctional problem of the theory of elasticity on the uniform rotation of a weightless ellipsoid made of a homogeneous isotropic material has been got within the limits of physical linearity and small.

2. Unlike the well-known more complex mathematical model [1, 2], the uncertainty related to the multivariance of the approximate solution of the problem [1, 2] by the inverse method [4, 8, 16] has been eliminated. In this regard, the proposed semi-inverse method [5, 14, 16] is mathematically accurate due to the direct (without intermediate relations (4) – (6)) assignment of two (23), (24) of the four desired stress functions in the form of the second degree polynomials with the complete exclusion of alternative (variant) in the analytical description of the stress state and the identical satisfaction of the system of initial equations (1), (2), (22), (40) together with boundary conditions (7), (8) on the surface of a rotating ellipsoidal body of rotation (Fig. 1).

3. The developed mechanical and mathematical model is brought to simple functional dependencies (34) – (37), (45) – (48) illustrated by a numerical example of definitions by formulas (54) – (58) when μ = 0.3 the main stresses in the characteristic cross sections of the sphere, which is a special case of an ellipsoid (Fig. 2).

4. A quantitative assessment of the approximate theory (14) – (19) [1, 2] has been carried out for three values of the Poisson 's ratio μ = 0; 0,3; 0,5, a quantitative assessment of the approximate theory in comparison with the new innovative solution has been carried out (38), (39) for the center r = z = 0 of the spherical body where extreme main stresses occur (Table 1 and 3). A sufficiently large discrepancy between the maximum internal force factors reaching 22.82% by stresses $\overline{\sigma}_r$ (0,0) = $\overline{\sigma}_\theta$(0,0) = $\overline{\sigma}$(0,0), σ_r (0,0) = σ_θ(0,0) = σ(0,0) and 89.93% - for the parameters $\overline{\sigma}_z$ (0.0), σ_z (0.0) has been established,

5. The conclusion part of the paper substantiates the possibility and relevance of the practical application of the derived formulas (34) – (37), (45) – (48) for a refined evaluation, based on the general equations of the volumetric stress-strain state [1, 2, 4, 33], the bearing capacity of elliptical discs (Fig. 3) used as critical elements of turbine impellers and turbochargers [27-30, 32-35].

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