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MECHANICS OF DEFORMABLE SOLID BODY

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A NEW TYPE OF BODY FORCE IN THE MECHANICS OF ELASTICALLY DEFORMABLE SOLIDS

СЕРПІМДІ ДЕФОРМАЦИЯЛАНАТЫН ҚАТТЫ ДЕНЕЛЕР МЕХАНИКАСЫНДАҒЫ ДЕНЕ КҮШІНІҢ ЖАҢА ТҮРІ

НОВЫЙ ВИД ОБЪЁМНОЙ СИЛЫ В МЕХАНИКЕ УПРУГОДЕФОРМИРУЕМОГО ТВЕРДОГО ТЕЛА

Abstract. *Based on the analysis and comparison of atomic physical and engineering models of solid matter, the existing classification of volumetric forces used in the resistance of materials and the theory of elasticity is expanded (magnetic, electromagnetic, inertial, gravitational). This fundamentally new addition is justified by replacing the force of interatomic interaction with a statistically averaged analogous equivalent in the form of a quasi-elastic volumetric load acting between infinitesimal material particles of a solid body which are displaced during its deformation. For continuous homogeneous isotropic materials obeying Hooke's law, using the fundamental law of conservation of mechanical energy, a general theory has been developed for determining the quasi-elastic body force function, the three projections of which* X_k, Y_k, Z_k *on the coordinate axes x, y, z linearly depend on the corresponding displacements u, v, w in x, y, z direction. The proposed calculation method is illustrated by the simplest example from the classical course of resistance of materials to uniaxial tension by external static forces P of a linearly elastic weightless beam of constant cross-section. In the course of a mathematically accurate solution to this innovative problem: 1) it was proven that the new quasi-elastic mass force depends on the location (coordinates) of an arbitrary point of the body, the external load acting on the structure, its volume and Poisson's ratio* μ, *and at* $\mu = 0{,}5$ *this volumetric load becomes zero; 2) the relevance and great importance of taking into account the quasi-elastic volumetric force in the equilibrium equations in the process of mathematical modeling of the stress-strain state and design of load-bearing structures has been confirmed.*

This approach opens up new opportunities for a more accurate assessment of the strength, rigidity, and stability of machine-building structural elements, including machine parts and mechanisms, automotive structural components, as well as aerospace and railway structures

Keywords: *force, stress, deformation, displacemen, homogeneity, isotropy, quasi-elasticity, surface and volume loads.*

Аңдатпа. *Қатты заттардың атомдық физикалық және инженерлік модельдерін талдау және салыстыру негізінде материалдардың кедергісінде қолданылатын көлемдік күштердің қолданыстағы классификациясы және серпімділік теориясы кеңейтілді: магниттік, электромагниттік, инерциялық, гравитациялық. Бұл түбегейлі жаңа қосу атомаралық әрекеттесу күшін оның деформациясы кезінде ығысатын қатты дененің шексіз аз материалдық бөлшектері арасында әрекет ететін квазисерпімді көлемдік жүктеме түріндегі статистикалық орташаланған аналогтық эквивалентпен алмастыру арқылы* *негізделеді. Гук заңына бағынатын үздіксіз біртекті изотропты материалдар үшін механикалық энергияның сақталу заңын қолдана отырып, дененің квазисерпімді күші функциясын анықтаудың жалпы теориясы жасалды, олардың үш проекциясы X_k, Y_k, Z_k координаталық осьтер бойынша x, y, z сәйкес орын ауыстыруларға сызықтық тәуелді u, v, w бағытта x, y, z. Ұсынылған есептеу әдісі тұрақты көлденең қимадағы сызықты серпімді салмақсыз арқалықтың P сыртқы статикалық күштерімен бір осьті керілуге материалдардың кедергісінің классикалық курсынан қарапайым мысалмен суреттелген. Осы инновациялық есептің математикалық нақты шешімі барысында: 1) жаңа квазисерпімді массалық күш дененің еркін нүктесінің орналасуына (координатасына), құрылымға әсер ететін сыртқы жүктемеге, оның көлеміне және Пуассон қатынасына тәуелді екендігі дәлелденді μ, және қашан $\mu = 0,5$ бұл көлемдік жүктеме нөлге айналады; 2) кернеулі-деформациялық күйді математикалық модельдеу және жүк көтергіш құрылымдарды жобалау процесінде тепе-теңдік теңдеулерінде квазисерпімді көлемдік күшті есепке алудың өзектілігі мен үлкен маңыздылығы расталды.*

Бұл тәсіл машиналар мен механизмдердің бөліктерін, автомобильдердің құрылымдық элементтерін, сондай-ақ ұшақтар мен темір жол жабдықтарының құрылымдарын қоса алғанда, машина жасау құрылымдарының элементтерінің беріктігін, қаттылығын және орнықтылығын неғұрлым дәл бағалау үшін жаңа мүмкіндіктер ашады.

Түйін сөздер: *күш, кернеу, деформация, қозғалу, біркелкілігі, изотропия, квази серпімділік, беттік және көлемдік жүктемелер.*

Аннотация. *Базируясь на анализе и сравнении атомарной физической и инженерно-технической моделей твердого вещества, расширена существующая классификация объемных сил, используемая в сопротивлении материалов и теории упругости: магнитных, электромагнитных, инерционных, гравитационных. Это принципиально новое дополнение обосновано заменой силы межатомного взаимодействия статистическим осредненным аналогичным эквивалентом в виде квазиупругой объемной нагрузки, действующей между бесконечно малыми материальными частицами твердого тела, которые смещаются при его деформации. Для сплошных однородных изотропных материалов, подчиняющихся закону Гука, с применением фундаментального закона сохранения механической энергии разработана общая теория определения функции квазиупругой объемной силы, три проекции которой X_k, Y_k, Z_k на координатные оси x, y, z линейно зависят от соответствующих перемещений u, v, w в направлении x, y, z. Предложенная методика расчета проиллюстрирована простейшим примером из классического курса сопротивления материалов на одноосное растяжение внешними статическими силами P линейно упругого невесомого бруса постоянного поперечного сечения. В ходе математически точного решения этой инновационной задачи: 1) доказано, что новая квази-упругая массовая сила зависит от места расположения (координат) произвольной точки тела, внешней нагрузки, действующей на конструкцию, её объема и коэффициента Пуассона μ, а при $\mu = 0,5$ данная объемная нагрузка становится равной нулю; 2) подтверждена актуальность и большая значимость учета в уравнениях равновесия квазиупругой объемной силы в процессе математического моделирования напряженно-деформированного состояния и проектирования несущих конструкций.*

Данный подход открывает новые возможности для более точной оценки прочности, жесткости и устойчивости элементов машиностроительных конструкций, включая детали машин и механизмов, конструктивные элементы автомобилей, а также конструкции авиационной и железнодорожной техники.

Ключевые слова: *сила, напряжение, деформация, перемещение, однородность, изотропность, квазиупругость, поверхностная и объемная нагрузки.*

Solid matter, in accordance with the idealized fundamental physical model, consists of interconnected and regularly located cubic cells, in the corners of which there are atoms at equal distances until the body deforms [Pavlov, Hohlov, et al., 2000; Gurevich, et al., 2004; Ashcroft, Cornell, et al., 2003; Patterson, et al., 2019; Hasbun, Datta, et al., 2019; Oma, et al., 1993]. In the elastic region, each atom is slightly displaced by a small amount in the direction opposite to the action of the restoring (quasi-elastic) point load (Figure 1)

$$F_K = -k \cdot x \tag{1}$$

with a proportionality coefficient, which is called the quasi-elastic force coefficient [Pavlov, Hohlov, et al., 2000; Gurevich, et al., 2004; Ashcroft, Cornell, et al., 2003; Patterson, et al., 2019; Hasbun, Datta, et al., 2019; Oma, et al., 1993; YAvorskij, Detlaf, et al., 1981; YAvorskij, Detlaf, et al., 1979].

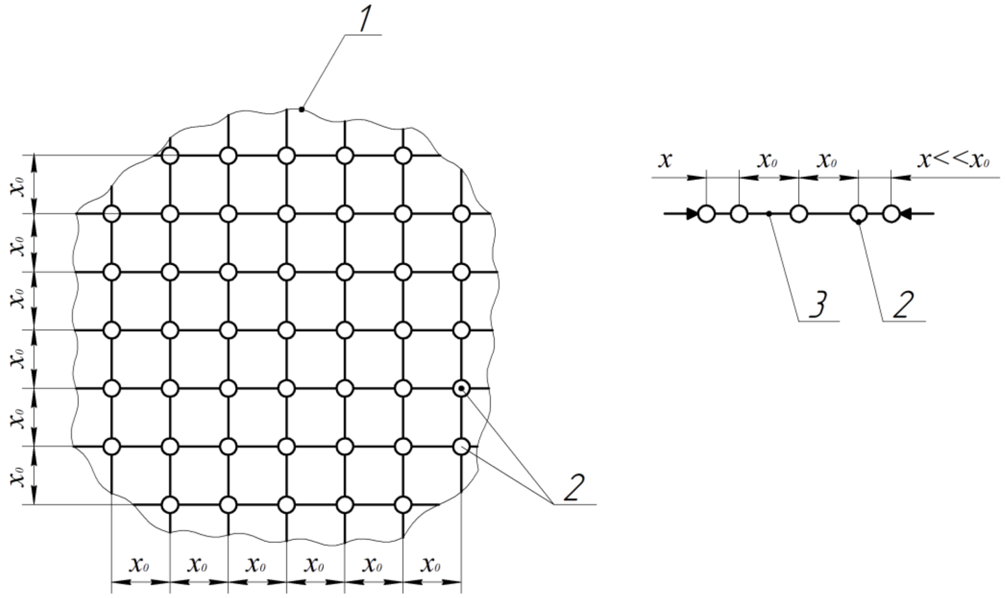

Figure 1. Interpretation on a plane of the ideal atomic structure of a solid from a physical point of view without taking into account imperfections (vacancies, interstitials, cracks, etc.): 1 – undeformed solid (material); 2 – atoms; 3 – diagram of the displacement of atoms during deformation of the body and the occurrence of restorative (quasi-elastic) $F_k = -k \cdot x$

Note – compiled by the authors

In engineering and technical modeling of the stress-strain state of a real load-bearing structure and its parts, used in applied mechanics of a deformable solid medium [Filin, et al., 1975; Timoshenko, Gere, et al., 1976], including the strength of materials [Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995] and the theory of elasticity [Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010], a homogeneous continuous isotropic solid substance (material) is represented in a rectangular frame of reference by an infinite number of material points having an elementary volume

$$dV = dx \cdot dy \cdot dz \,. \tag{2}$$

Each point $B(x, y, z)$ in the case of deformation of the body moves by a certain amount

$$BB_1 = BB_1(x, y, z) \tag{3}$$

with corresponding projections

$$u = u(x, y, z),\ \upsilon = \upsilon(x, y, z),\ w = w(x, y, z) \tag{4}$$

to coordinate axes (x, y, z) [Filin, et al., 1975; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995; Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010] (Figure 2). Kinematic parameters $u, \upsilon, w > 0$ if their directions coincide with $x, y, z > 0$, as in Figure 2.

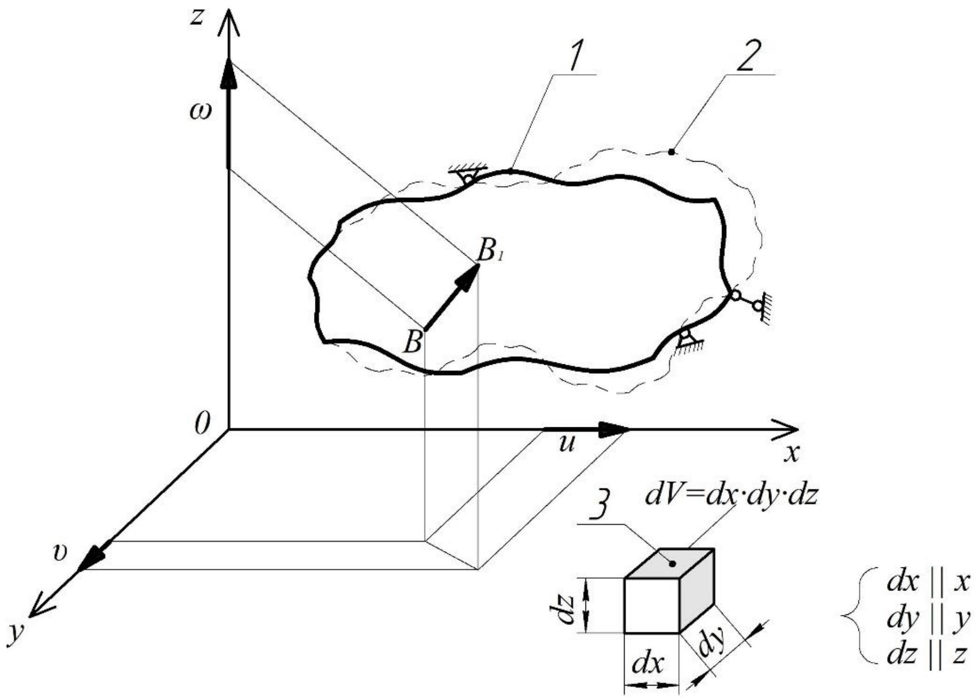

Figure 2. Scheme of elastic displacement of a material point $B(x, y, z)$ by the amount BB_1 : 1 – rigid body fixed to deformation; 2 – elastically deformed state of the body; 3 – elementary parallelepiped in the vicinity of any arbitrary point of the elastic medium
Note – compiled by the authors

Comparing the atomic model of a solid (Figure 1) with the engineering model (Figure 2), one can note their obvious physical and mechanical analogy. In the first case (Figure 1), due to elastic deformation of the body, atoms are displaced under the influence of external ones, and in the second (Figure 2) material points move. On this basis, it is permissible, according to (1), to introduce into the equations of equilibrium (statics) and motion (dynamics) of an elastically deformable solid body, as projections on the axis $x, y, z > 0$, additional volumetric quasi-elastic or restoring loads

$$X_k = X_{K(x,y,z)} = -k \cdot u \,,\; Y_k = Y_{K(x,y,z)} = -k \cdot \upsilon \,,\; Z_k = Z_{K(x,y,z)} = -k \cdot w \,, \tag{5}$$

in addition to the already known components X, Y, Z [Filin, et al., Timoshenko, Gere, et al., 1976; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995]. Naturally, the coefficient in linear relations (5) must have the dimension $P \cdot L^{-4}$ (P – force, L – length).

We also argue for the implementation of the addition procedure (5) by the impossibility of determining the force (1) of interaction between each pair of atoms (Figure 1) and, in connection with this, the need to replace the volumetric quasi-elastic loads (5) arising between infinitesimal material particles of solid matter with an averaged statistical equivalent (Figure 2).

As for the numerical determination of the parameter in force projections (5) and its mechanical and mathematical content, then, at least for physically linear, homogeneous, isotropic, solid materials, this problem in the static formulation, as will be shown below, is easily solved analytically by energy method.

It is known [Pavlov, Hohlov, et al., 2000; Gurevich, et al., 2004; Ashcroft, Cornell, et al., 2003; Patterson, et al., 2019; Hasbun, Datta, et al., 2019; Oma, et al., 1993; YAvorskij, Detlaf, et al., 1981; YAvorskij, Detlaf, et al., 1979; Filin, et al., 1975; Timoshenko, Gere, et al., 1976; Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010; Voronkov, et al., 1961; Belyaev, et al., 1965; Strelkov, et al., 1975]. that at stresses not exceeding the elastic limit, the change in the thermal and electromagnetic state is insignificant and can be neglected. As a result, all the work A of external forces, based on the law of conservation of mechanical energy [Filin, et al., 1975; Feodos'ev, et al., 2001; Voronkov, et al., 1961; Belyaev, et al., 1965; Strelkov, et al., 1975], accumulates in the body material in the form of the sum of potential U and kinetic K energies of deformation, that is

$$U + K = A, \tag{6}$$

where in the case of a static stress-strain state of the structure

$$K = 0. \tag{7}$$

When determining A, we take into account [Filin, et al., 1975; Timoshenko, Gere, et al., 1976; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995; Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010; Belyaev, et al., 1965], both volumetric $X + X_k$, $Y + Y_k$, $Z + Z_k$ and surface loads X_γ, Y_γ, Z_γ, and express the energy component of equality (6) through the principal normal stresses

$$\sigma_1 = \sigma_1(x, y, z),\ \sigma_2 = \sigma_2(x, y, z),\ \sigma_3 = \sigma_3(x, y, z),\ \sigma_1 > \sigma_2 > \sigma_3, \tag{8}$$

acting along three mutually perpendicular elementary areas in the vicinity of an arbitrary point B (Figure 2) [Filin, et al., 1975; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Belyaev, et al., 1965]:

$$\begin{aligned} U = U_0 + U_f &= \frac{1-2\mu}{6E}\int_V (\sigma_1 + \sigma_2 + \sigma_3)^2 dV + \\ &+ \frac{1+\mu}{3E}\int_V \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right) dV = \frac{1}{2E}\int_V \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] dV; \end{aligned} \tag{9}$$

where U_0, U_f, – potential energies of change in volume (0) and shape (f) of the body (parts, structures)

$$U_0 = \frac{1-2\mu}{6E}\int_V (\sigma_1 + \sigma_2 + \sigma_3)^2 \cdot dV, \tag{10}$$

$$U_f = \frac{1+\mu}{3E}\int_V \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right) dV; \tag{11}$$

E, μ – respectively, the elastic modulus and Poisson's ratio of the material.

Let us approximate the right side of the energy balance (6) by the sum of two components

$$A = A_p + A_k, \tag{12}$$

under conditions (5), (7) and compliance with Hooke's law, in which [Filin, et al., 1975; Timoshenko, Gere, et al., 1976; Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Timoshenko, et al., 1975]: A_p – work of external surface X_γ, Y_γ, Z_γ and volumetric X, Y, Z loads, with the exception of (5) and inertial force factors (static problem),

$$A_p = \frac{1}{2}\left[\int_V \left(X \cdot u + Y \cdot \upsilon + Z \cdot w\right)dV + \int_S \left(X_\gamma \cdot u + Y_\gamma \cdot \upsilon + Z_\gamma \cdot w\right)dS\right], \qquad (13)$$

$A_K \le 0$ – total work from projections (5) on the axis of quasi-elastic mass forces X_K, Y_K, Z_K

$$A_p = \frac{1}{2}\left[\int_V \left(X \cdot u + Y \cdot \upsilon + Z \cdot w\right)dV + \int_S \left(X_\gamma \cdot u + Y_\gamma \cdot \upsilon + Z_\gamma \cdot w\right)dS\right], \qquad (14)$$

V – volume (V) and area (S) of the body (structure).

Taking into account (7), (9)-(12), we present the law of energy constancy (6), when $K = 0$:

$$U_0 + U_f = -A + A_p + A + A_k = -A_k + 2A_k + A_p, \quad \begin{cases} -A + A_p = -A_k \\ A + A_k = 2A_k + A_p. \end{cases} \qquad (15)$$

Using a specific example, taking into account (10), (11), (13), (14) (see below) two combination options U_0, U_f, A_k, A_p,

$$1) \begin{cases} U_0 = -A_k, & (16) \\ U_f = 2A_k + A_p & (17) \end{cases}$$

$$2) \begin{cases} U_0 = 2A_k + A_p, & (18) \\ U_f = -A_k & (19) \end{cases}$$

a rigorous proof has been found that only if conditions (16), (17) are used, the required and physically and mathematically justified result is obtained when determining the coefficient $k \ge 0$ quasi-elastic body forces (5).

As a typical simple example from the classical course on strength of materials [Timoshenko, Gere, et al., 1976; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995; Belyaev, et al., 1965], confirming the need and relevance of taking into account X_k, Y_k, Z_k, let us consider uniaxial tension by external static forces a linearly elastic weightless beam constant along its length cross section (Figure 3).

The diagrams in Figure 3 show a typical solution for the resistance of materials for stresses [Filin, et al., 1975; Timoshenko, Gere, et al., 1976; Ickovich, et al., 2001; Feodos'ev, et al., 2001; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995; Belyaev, et al., 1965]

$$\sigma_s = \frac{P}{F} = const, \Rightarrow \sigma_s^* = \sigma_s \cdot \frac{F}{P} = 1 \qquad (20)$$

marked with the index "s", and the «asterisk» icon indicates the corresponding dimensionless parameter $\sigma_s^* = 1$. The formula for absolute linear deformation u_s looks like a simple linear function

$$u_s = u_s^{(x)} = \frac{P \cdot x}{EF}, \Rightarrow u_s^* = u_s \cdot \frac{EF}{Pl} = \frac{x}{l}, \; -l \le x \le l. \qquad (21)$$

For a shaded infinite small beam element with volume (Figure 3)

$$dV = F \cdot dx \tag{22}$$

condition of static equilibrium in the form of the sum of projections of all forces onto the axis x,

$$\sum x = 0, \tag{23}$$

at $X = Y = Z = Y_k = Z_k = 0$, taking into account (5) and (22), has the form:

$$-\sigma \cdot F + (\sigma + d\sigma) \cdot F + X_k dV = -\sigma F + (\sigma + d\sigma) \cdot F - k \cdot u \cdot F \cdot dx = 0, \Rightarrow \tag{24}$$

$$\frac{d\sigma}{dx} - k \cdot u = 0. \tag{25}$$

Let's imagine the relative linear deformation ε in Hooke's law

$$\sigma = E \cdot \varepsilon = E \cdot \frac{du}{dx} \tag{26}$$

differential geometric Cauchy dependence [Filin, et al., 1975; Timoshenko, Gere, et al., 1976; Vardanyan, Andreev, Atarov, Gorshkov et al., 1995; Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010]

$$\varepsilon = \frac{du}{dx}. \tag{27}$$

Substituting (26) into (25), we will have a homogeneous linear differential equation of the second order [Bronshtejn, Semendyaev, et al., 1986; Kamke, et al., 1976]

$$\frac{d^2u}{dx^2} - \beta^2 \cdot u = 0 \tag{28}$$

with a constant coefficient

$$\beta^2 = \frac{k}{E}. \tag{29}$$

In the hyperbolic functions of sine $sh\beta x$ and cosine $ch\beta x$ general solution [Kamke, et al., 1976]

$$u = u(x) = C_1 sh\beta x + C_2 \cdot ch\beta x \tag{30}$$

equation (28) contains two arbitrary integration constants C_1 and C_2, which we find from the boundary conditions (Figure 3):

– for a fixed point *O*, due to the symmetry of the calculation model, where $x = 0$,

$$u(0) = 0, \Rightarrow C_2 = 0; \tag{31}$$

in extreme cross sections $x = \pm l$, using (20), (26), (30), (31),

$$\sigma = \sigma(x) = E\frac{du}{dx} = E \cdot C_1 \cdot \beta \cdot ch\beta x, \Rightarrow \tag{32}$$

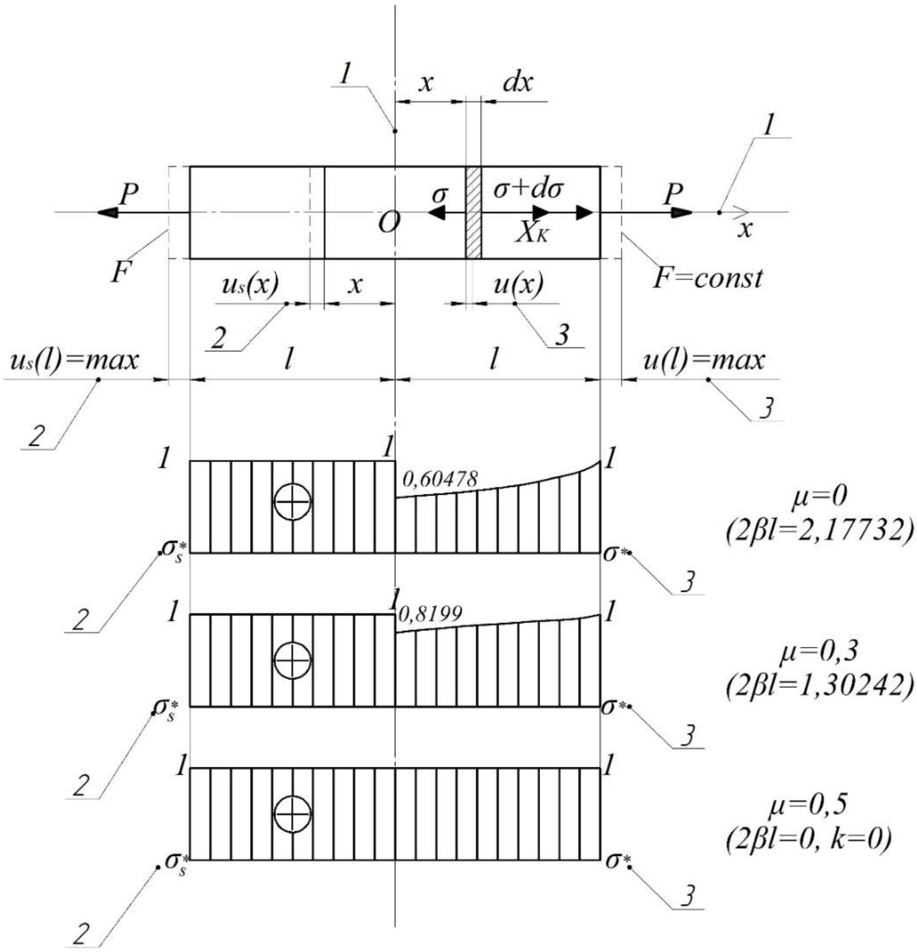

Figure 3. Refined force X_k calculation diagram of the rod and dimensionless epiures of normal stresses: 1 – axis of symmetry; 2 – elementary solution according to the formulas for the strength of materials (20), (21); 3 – new solution of the problem obtained in accordance with (34) – (36) taking into account the quasi-elastic volumetric load

Note – compiled by the authors

$$\sigma(\pm l) = EC_1 \cdot \beta \cdot ch\beta l = \frac{P}{F}, \Rightarrow C_1 = \frac{Pl}{EF \cdot \beta l \cdot ch\beta l} \tag{33}$$

Taking into account (31), (33), we obtain from (30), (32) two final formulas:

$$\begin{cases} u = u(x) = \dfrac{Pl}{EF \cdot \beta l} \cdot \dfrac{sh\beta x}{ch\beta l}; & (34) \\ \sigma = \sigma(x) = \dfrac{P}{F} \cdot \dfrac{ch\beta x}{ch\beta l}, -l \le x \le l. & (35) \end{cases}$$

It is quite obvious that at $\beta = 0\ (\Rightarrow k = 0)$, according to (29), the derived analytical relations (34), (35) are transformed into simplified elementary dependencies (20), (21). Naturally, the same result of transforming (34), (35) into (20), (21) will be obtained directly from the differential equation (28) and Hooke’s law (26), when $\beta = 0$.

The dimensionless form of the refined solution (35) for normal stress σ is represented as follows (Figure 3):

$$\sigma^* = \sigma^*(x) = \sigma\frac{F}{P} = \frac{ch\beta x}{ch\beta l},\ -l \le x \le l\,. \tag{36}$$

From the final formulas (20), (21) and (34), (35) it follows that, in contrast to the u_s, σ_s, obtained new characteristics u, σ, the stress-strain state of the rod has a pronounced nonlinear nature due to taking into account the volumetric quasi-elastic force $X_k = -k \cdot u$.

Continuing to solve the specific problem posed (Figure 3), we determine the potential energies U, U_0, U_f and work A, A_p, A_f for the right half of the beam, due to the symmetry of its design scheme. For this purpose, we will use general formulas (5), (9) - (11), (12) - (14), (29), (34), (35) and Table integrals [Smolyanskij, et al., 1965]

$$\left.\begin{aligned} \int ch^2 \beta x dx &= \frac{sh2\beta x}{4\beta} + \frac{x}{2}, \\ \int sh^2 \beta x dx &= \frac{sh2\beta x}{4\beta} - \frac{x}{2}, \end{aligned}\right\} \tag{37}$$

taking in this particular case: $\sigma_1 = \sigma,\quad \sigma_2 = \sigma_3 = 0$; $X = Y = Z = 0$; $Y_\gamma = Z_\gamma = 0$; $\upsilon = w = 0, \Rightarrow Y_k = Z_k = 0$ (see (5)); $dV = F \cdot dx$;

$$A_p = \int_S X_\gamma \cdot u \cdot ds = \frac{P \cdot u(l)}{2}. \tag{38}$$

As a result, we get $F = const$:

$$U = \frac{F}{2E}\int_0^l \sigma^2 dx = \frac{P^2 l}{4EF \cdot ch^2\beta l} \cdot \left(1 + \frac{sh2\beta l}{2\beta l}\right) \ge 0, \tag{39}$$

$$U_0 = \frac{(1-2\mu) \cdot F}{6E}\int_0^l \sigma^2 \cdot dx = \frac{1-2\mu}{3} \cdot U \ge 0, \tag{40}$$

$$U_f = \frac{(1+\mu) \cdot F}{3E}\int_0^l \sigma^2 \cdot dx = \frac{2 \cdot (1+\mu)}{3} \cdot U \ge 0; \tag{41}$$

$$A_p = \frac{P \cdot u(l)}{2} = \frac{P^2 l}{2EF \cdot \beta l} \cdot \frac{sh\beta l}{ch\beta l} = \frac{P^2 l \cdot sh2\beta l}{2EF \cdot 2\beta l \cdot ch^2\beta l} \ge 0, \tag{42}$$

$$A_k = -\frac{k \cdot F}{2}\int_0^l u^2 \cdot dx = -\frac{-\beta^2 EF}{2}\int_0^l u^2 \cdot dx = \frac{P^2 l}{4EF \cdot ch^2\beta l} \cdot \left(1 - \frac{sh2\beta l}{2\beta l}\right) \le 0, \tag{43}$$

$$A = A_k + A_p = \frac{P^2 l}{4EF \cdot ch^2\beta l} \cdot \left(1 + \frac{sh2\beta l}{2\beta l}\right) = U. \tag{44}$$

Using expressions (40) - (43), we check the energy balances (16), (17) and (18), (19) in order to determine the non-zero value of the dimensionless parameter $2\beta l$, and therefore, according to (29), and coefficient $k \ge 0$ volumetric load X_k.

As a result of implementing this procedure:

1) from conditions (16), (17) one transcendental equation is derived [Bronshtejn, Semendyaev,

et al., 1986]

$$sh2\beta l = \frac{2-\mu}{1+\mu} \cdot 2\beta l\,, \tag{45}$$

the desired solution of which $0 \le 2\beta l \le 2{,}17732 = \max$ was carried out by the selection method for six values of the coefficient μ, covering the entire permissible range $0 \le \mu \le 0{,}5$, and is presented in Figure 4 by a graph of the dependence $2\beta l = 2\beta l(\mu)$, constructed from the numerical data of Table 1.

2) the disclosure of equalities (18), (19) leads, accordingly, to two similar, but not identical transcendental relations

$$\begin{cases} sh2\beta l = \dfrac{5+2\mu}{1-2\mu} 2\beta l, & (46) \\ sh2\beta l = 2\beta l, & (47) \end{cases}$$

the simultaneous (joint) existence of which is possible only in the case $2\beta l = 0 (\Rightarrow k = 0)$.

Table 1. Calculation information for solving equations (45)

μ	0	0,1	0,2	0,3	0,4	0,5
$2\beta l$	2,17732	1,90802	1,62212	1,30242	0,907	0
Note – compiled by the authors						

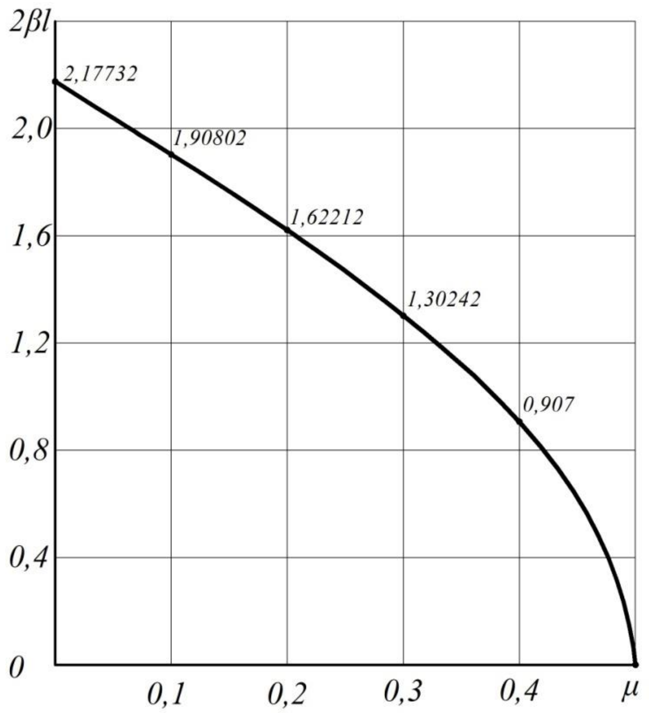

Figure 4. Graphical representation of functions for solving equations (45)

Note – compiled by the authors

Analyzing the results of the calculation $2\beta l \ge 0$, it can be argued that from a mechanical and mathematical point of view, there are only two joint energy combinations (16) and (17), proving the existence of quasi-elastic volumetric forces (5).

Moreover, for such soft materials as resin and rubber, which have $\mu \approx 0{,}5$ - [Belyaev, et al., 1965; Lavendel, et al., 1976; Anur'ev, et al., 2006], the volumetric load X_k is practically absent, and with Poisson's ratio $\mu = 0$ for cork (natural material) [Belyaev, et al., 1965] value $2\beta l = 2{,}17732 = \max$, from which it follows $X_k = \max$, $k = \max$, (see (5) and (29)).

Complementing this analysis, it should be noted that if $\mu = 0{,}5$, then a change in the volume of the body and its potential energy U_0, according to (10), (40), does not occur, which is possible and actually confirmed only for practically incompressible liquid substances, located in tanks under excess pressure [YAvorskij, Detlaf, et al., 1981; YAvorskij, Detlaf, et al., 1979].

The calculation of the dimensionless stress diagrams $\sigma^* = \sigma^*(x)$ shown in Figure 3 was carried out using formula (36) and quantitative information presented in tabular form (see Tables 1 and 2).

It is of considerable interest to compare extreme values $\sigma^*_{\min} = \sigma^*(0), \sigma^*_s = 1$, and

$$u_s^*(l) = \frac{x}{l} = 1 = \max, \tag{48}$$

$$u^*(l) = \frac{th\beta l}{\beta l} = \max, \tag{49}$$

using for this purpose the diagrams of Figure 3, relations (21), (34) at $x = l$ and Tables 1, 2, 3.

Table 2. Numerical data about the function (36)

μ	βl	x	0	$\pm 0{,}2l$	$\pm 0{,}4l$	$\pm 0{,}6l$	$\pm 0{,}8l$	$\pm l$
$\mu = 0$	1,08866	$\sigma^*(x)$	0,60478	0,61918	0,66304	0,73846	0,84902	1
$\mu = 0{,}3$	0,65121	$\sigma^*(x)$	0,8199	0,82688	0,84789	0,8833	0,93372	1
$\mu = 0{,}5$	$\beta l = 0$	$\sigma^*(x)$	1	1	1	1	1	1
Note – compiled by the authors								

Table 3. Estimation in percentage of errors Δ_u, Δ_σ, approximate formulas (20), (21) (or (48)) in relation to the refined dependencies (36), (49).

μ (βl)	0 (1,08866)	0,3 (0,65121)	0,5 (0)
$\Delta_u = \dfrac{u^*(l) - u_s^*(l)}{u_s^*(l)} \cdot 100\%$	-26,85	-12,09	0
$\Delta_\sigma^* = \dfrac{\sigma^*(0) - \sigma_s^*}{\sigma_s^*}$	-39,52	-18,01	0
Note – compiled by the authors			

Concluding our consideration of the example (Figure 3), we present a formula that allows us to quantify the magnitude of the body force X_k for any Poisson's ratio $0 \le \mu \le 0{,}5$.

Let us first designate the value of the dimensionless product $2\beta l$ found from equation (45) as a functional parameter $\zeta = \zeta(\mu)$, depending on μ, that is (see Table 1)

$$2\beta l = \zeta = \zeta(\mu) \tag{50}$$

Replacing the constant β in equality (50) in accordance with (29), we find the coefficient k:

$$2l\sqrt{\frac{k}{E}} = \zeta, \Rightarrow k = \frac{\zeta \cdot E}{4l^2}. \tag{51}$$

Substituting (51) and displacement u, according to relation (34), into the first formula (5), we determine the volumetric quasi-elastic force

$$\begin{aligned} X_k = X_k(x) = -k \cdot u = -\frac{\zeta \cdot E}{4l^2} \cdot \frac{Pl}{EF \cdot \beta l} \cdot \frac{sh\beta x}{ch\beta l} = \\ = -\frac{\zeta \cdot P}{2l \cdot F\zeta} \cdot \frac{sh\beta x}{ch0{,}5\zeta} = -\frac{P}{V} \cdot \frac{sh0{,}5\zeta \cdot \dfrac{x}{l}}{ch0{,}5\zeta}, \end{aligned} \tag{52}$$

depending on the tensile external load P, the volume of the rod $V = 2Fl$ (Figure 3), Poisson's ratio μ and the variable x, changing for the right half of the beam in the interval $0 \le x \le l$.

The nature of the found increasing hyperbolic function [Bronshtejn, Semendyaev, et al., 1986] $X_k(x)$ indicates that its maximum will be at the extreme point of the rod $x = l$

$$X_k(l) = -\frac{P}{V} th0{,}5\zeta = \max; \tag{53}$$

where the tangent $th0{,}5\zeta$ varies within (see Table 1):

- from $th0{,}5 \cdot 2{,}17732 = 0{,}79639$ to $\mu = 0$,
- up to $th0 = 0$ at $\mu = 0{,}5$.

The derived formulas (52), (53) show that quasi-elastic mass forces arise in any stressed structure (part), while the known volumetric loads - magnetic, electromagnetic, inertial, gravitational - may not be associated with the deformed state of the material.

An indirect sign of the presence and influence of the quasi-elastic force X_k (Figure 1) are educational laboratory experiments on tensile steel ($\mu = 0{,}3$) standard samples, when the point of rupture of the rod: 1) almost always approaches its movable end fastening in a special grip of the test bench machine, that is, to the most stressed section of the beam with $\sigma(l) = \sigma_{\max}$, $X_k(l) = \max$; 2) never coincides with the fixed point $x = 0$, where the displacement is $u(0) = 0$ (see (34), Figure 3).

Conclusions:

1) Based on a reasoned conceptual analogy between the ideal physical and engineering models of a solid body (Figure 1,2), it is proposed to supplement the equilibrium equations of the theory of elasticity and resistance of materials with fundamentally new parameters in the form of three projections (5) onto the coordinate axes x,y ,z volumetric quasi-elastic force applied to an arbitrary material point.

2) Using the law of conservation of mechanical energy (6), a general computational and

theoretical framework has been developed that makes it possible to determine the quasi-elastic volumetric load function for continuous homogeneous isotropic materials that obey Hooke's law.

3) A simple and illustrative example from a typical classical course on strength of materials about uniaxial tension by external static forces P of a linearly elastic weightless rod of constant cross-section is considered (Figure 3). The solution to this problem is brought to the final formulas (16), (17), (29), (34), (35), (45), (49) – (53) and illustrated with diagrams of normal stresses for three values of the Passon coefficient μ = 0; 0,3; 0,5 (Figure 3). While analyzing the results of this example:

• the dependence of the new body force, according to (52), (53), on the location (coordinates) of an arbitrary material point of the body, the volume of the structure (part) and the transverse deformation coefficient has been proven μ and in the case of μ = 0,5, this mass quasi-elastic force becomes equal to zero; from here it follows, guided by (16), (17), (45), the absence of volumetric potential energy $V_0 = 0$ and non-zero value $V_f \geq 0$ due to a change in the shape of the beam deformed by the load *P*;

• a significant (and increasing with decreasing μ) error in the elementary linear formulas (20), (21) of the resistance of materials was established in relation to the obtained refined-modified solution (34), (35), approximated by hyperbolic functional curves (see Figure 3 and Table 2, 3);

• the relevance and great practical significance of taking into account the quasi-elastic volumetric force in the equilibrium equations in the mathematical modeling of the stress-strain state of load-bearing structures and their elements has been confirmed.

Concluding the article and analytically summarizing the overall results of the research work carried out, we can formulate some of the most important considerations and prerequisites on this problematic issue.

Taking into account quasi-elastic volumetric forces X_k , Y_k, Z_k, depending linearly on the corresponding displacements u, v, w (see (5) and Figure 2) involves abandoning the use of the classical concept of static definability of the designed structure when determining stresses, since all mechanical systems in this case become internally statically indeterminate. Of course, this statement does not apply to the external features of the design design, which presuppose the presence of connections (supports), the number of which is equal to the number of equations of equilibrium or motion (in dynamics) to determine the support reactions.

The introduction of the concept of quasi-elastic volumetric load into the applied mechanics of a deformable solid significantly refines the design scheme of a real structure, which makes it possible to identify a number of deep phenomena, properties and features of the designed object that do not appear when using known physical and mathematical models.

From this point of view, a typical example can be the problem of clarifying the fundamental solution of the French mathematician and engineer Gabriel Lamé [Lame, et al., 1852] on the stressed axisymmetric state of an elastic solid cylinder. The authors of article [Abdeev, Muslimanova, et al., 2012] proved that the Lamé model: is not correct and adequate for Poisson's ratio μ < 0,5; contradicts the uniqueness theorem for solving problems in the linear theory of elasticity [Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Timoshenko, et al., 1975; Lur'e, et al., 1970; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010]; does not agree with the principle of locality of the self-balanced load effect (Saint-Venant's principle [Han, et al., 1988; Timoshenko, et al., 1975; Lur'e, et al., 1970; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002]). The paradox noted in [Abdeev, Muslimanova, et al., 2012] is eliminated by supplementing the statics equation in the Lamé mathematical model [Lame, et al., 1852] with a radial quasi-elastic body force, but this problem is beyond the scope of this article.

The same calculation scheme of uniaxial tension (Figure 3), but taking into account transverse deformations, is of undoubted scientific, theoretical and practical interest. The consequence of this premise is the introduction of additional projections into the rod element modeled in Figure 3 Y_k, Z_k a quasi-elastic volumetric force (see (5)) and the transformation of a one-dimensional model of the resistance of materials into a spatial one, including 15 sought-after mechanical-geometric characteristics of the stress-strain state. And this is a mathematical problem of the linear theory of elasticity [Struzhanov, Burnasheva, et al., 2019; Han, et al., 1988; Kac, et al., 2002; Skripnyak, ZHukova, Skripnyak, et al., 2005; Timoshenko, et al., 1975; Lur'e, et al., 1970; Sapunov, et al., 2011; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Sapunov, et al., 2008; Dudyak, Sahnovich, et al., 2010], which can be significantly simplified to two-dimensional for a beam of circular cross-section, transforming it (the problem) using rectangular coordinates x, r a cylindrical reference system $xr\theta$ into an axisymmetric one, adding two static equilibrium equations (see (5)) with axial X_k = -k·u and radial R_k = -k·w mass restoring (quasi-elastic) loads.

The scientific, technical and methodological aspects of the problem raised are so extensive and multifaceted that they do not allow us to note in this textual material at least the main directions for further promising research. First of all, these can be problems on simple types of resistance of materials, but taking into account the influence of quasi-elastic volumetric forces, including in combination with temperature, inertial loads and, accordingly, kinetic energy. Secondly, the issues of a more precise assessment of the strength, rigidity and stability of structures seem relevant. Here, there may be real prospects for reducing standard safety factors, due to the improvement of mechanical and mathematical models and, ultimately, increasing the reliability and efficiency of structures. Of significant interest in this regard are structural nonlinear [Lukash, et al., 1978], as well as related contact problems [Filin, et al., 1975; Han, et al., 1988; Kac, et al., 2002; Timoshenko, et al., 1975; Lur'e, et al., 1970; Gorshkov, Starovojtov, Tarlakovskij, et al., 2002; Dudyak, Sahnovich, et al., 2010], which are widely used in construction and mechanical engineering [Birger, SHorr, Iosilevich, et al., 1979].

References

Abdeev B.M., Muslimanova G. (2012). Protivorechie v osesimmetrichnoj klassicheskoj zadache Lame o napryazhennom sostoyanii uprugogo cilindra / Sb. trudov IX MNPK «Tekhnicheskie nauki – ot teorii k praktike» (17 aprelya 2012 g.). – Novosibirsk (Rossiya): Izd-vo «Sibirskaya associaciya konsul'tantov», s. 46-50.

Anur'ev V.I. (2006). Spravochnik konstruktora-mashinostroitelya: v 3-h t., T. 1 – 9-e izd., pererab. i dop. /pod red. I.N. ZHestkovoj. – M.: Mashinostroenie, 928 s.

Ashcroft N.W., David Cornell N. (2003). Solid State Physics: Cengage Learning (USA), 2003. – 832 p.

Belyaev N.M. (1965). Soprotivlenie materialov. – M.: Nauka, 856 s.

Birger I.A., SHorr B.F., Iosilevich G.B. (1979). Raschet na prochnost' detalej mashin: Spravochnik / 3-e izd., pererab. i dop. – M.: Mashinostroenie, 702 s.

Bronshtejn I.N., Semendyaev K.A. (1986). Spravochnik po matematike dlya inzhenerov i uchashchihsya vtuzov. 13-e izd. ispr. – M.: Nauka, 344 s.

Dudyak A.I., Sahnovich T.A. (2010). Prikladnaya teoriya uprugosti. – Minsk: Izd-vo Grevcova, 164 s.

Feodos'ev V.I. (2001). Soprotivlenie materialov [Tekst]: Uchebnik. – M.: Izd-vo MGTU im. Baumana, 591 s.

Filin A.P. (1975). Prikladnaya mekhanika tverdogo deformiruemogo tela. T. 1. – M.: Nauka, 832 s.

Gorshkov A.G., Starovojtov E.I., Tarlakovskij D.V. (2002). Teoriya uprugosti i plastichnosti: Uchebnik dlya vuzov. – M.: FIZMATLIT, 416 s.

Gurevich A.G. (2004). Fizika tverdogo tela: Uchebnoe posobie dlya vuzov /FTI im. A.F. Ioffe RAN. – SPb.: Nevskij dialekt; BHV – Peterburg, 320 s.

Han H. (1988). Teoriya uprugosti. Osnovy linejnoj teorii i eyo prilozheniya /per. s nem. – M.: Mir, 344 s.

Hasbun J.E., Datta T. (2009). Introductory Solid State Physics with MATLAB Applicathion, CPC Press (USA). 550 p.

Ickovich G.M. (2001). Soprotivlenie materialov: Uchebnik. – M.: Vysshaya shkola, 363 s.

Kac A.M. (2002). Teoriya uprugosti. – SPb: Izd-vo Lan', 208 s.

Kamke E. (1976). Spravochnik po obyknovennym differencial'nym uravneniyam /per. s nem. S.V. Fomina. – M.: Nauka, 576 s.

Lame G. (1852). Lecons sur la theorie matematique de I'elastisite des corps solides. Paris, Bachelier, 335

p.
Lavendel E.G. (1976). Raschet rezinotekhnicheskih izdelij. – M.: Mashinostroenie, 232 s.
Lukash P.A. (1978). Osnovy nelinejnoj stroitel'noj mekhaniki. – M.: Strojizdat, 204 s.
Lur'e A.I. (1970). Teoriya uprugosti. – M.: Nauka, 945 s.
Omar A. (1993). Elementary Solid State Physics: Principles and Applications, Pearson (USA), 600 p.
Patterson J.D. (2019). Solid State Physics: Introduction to the Theory, Spinger, B.C. Bailey (USA), 79 p.
Pavlov A.P., Hohlov A.F. (2000). Fizika tverdogo tela: Uchebnik. – 3-e izd., ster. – M.: Vysshaya shkola, 484 s.
Sapunov V. T. (2011). Zadachi prikladnoj teorii uprugosti: Uchebnoe posobie dlya vuzov. – M.: Izd-vo Nac. issled. yader. un-ta «Moskovskij inzhenerno-fizicheskij institut», 208 s.
Sapunov V.T. (2008). Prikladnaya teoriya uprugosti: v 2 ch. – M.: Izd-vo MIFI, 232 s.
Skripnyak E.G., ZHukova T.V., Skripnyak V.A. (2005). Matematicheskaya postanovka zadach linejnoj teorii uprugosti: Uchebnoe posobie. – Tomsk: Izd-vo TGU, 26 s.
Smolyanskij M.L. (1965). Tablicy neopredelennyh integralov. – M.: Nauka, 112 s.
Strelkov S.P. (1975). Mekhanika: Ucheb. posobie dlya universitetov. – M.: Nauka, 600 s.
Struzhanov V.V., Burnasheva N.V. (2019). Teoriya uprugosti: osnovnye polozheniya: Uchebnoe posobie. – Ekaterinburg: Izd-vo Ural'skogo un-ta, 204 s.
Timoshenko S.P., Dzh. Gere. (1976). Mekhanika materialov /per s angl. A.G. Kornejchuka, pod red. E.I. Grigolyuka. – M.: Mir, 672 s.
Timoshenko S.P., Gud"er Dzh. (1975). Teoriya uprugosti /per. s angl. M.I. Rejtmana, pod red. G.S. SHapiro. – M.: Nauka, 576 s.
Vardanyan G.S., Andreev V.I., Atarov N.M., Gorshkov A.A. (1995). Soprotivlenie materialov s osnovami teorii uprugosti i plastichnosti. – M.: Izd-vo Associacii stroitel'nyh vuzov (ASV), 568 s.
Voronkov I.M. (1961). Kurs teoreticheskoj mekhaniki. – M.: Gos. izd-vo fiziko-matematicheskoj literatury, 596 s.
YAvorskij B.M., Detlaf A.A. (1981). Spravochnik po fizike. – M.: Nauka, 512 s.
YAvorskij B.M., Detlaf A.A. (1979). Spravochnik po fizike: dlya inzhenerov i studentov vuzov. – M.: Nauka, 944 s.

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